These measured unsteady effects, in particular the measurements of amplitude of oscillation, provide fundamental data for evaluation of unsteady shear flow models. This work is presently being done by the authors.

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Numerical Solutions of Transonic Flows by Parametric Differentiation and Integral Equation Techniques

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Introduction

RECENT experimental and theoretical investigations of external transonic flows over aerodynamic elements have as their goal the prediction of such flows with greater accuracy and efficiency. Tijdeman and Schippers[1] and Davis and Malcolm[2] have provided an excellent set of experimental data to assist in the development of analytical models of transonic flows. The theoretical research programs may be considered chiefly under three classifications: finite-difference approximations,[3] finite-element approximations,[4] and integral equation methods.[5-10] The incentive for using integral equation methods (IEM) over conventional finite-difference approaches lies in the apparent relative easiness in numerically solving the integral equation.

Recent advances in IEM's have resulted in predictions of both lifting and nonlifting cases including supercritical flows. The results obtained by Nixon[10] represent a significant advance in IEM development. In addition to considering first-order corrections for shock wave curvature, Nixon[10] has eliminated the simplification that the perturbation velocity is zero upstream and downstream of the airfoil. Independently, we have developed an IEM which is free of this simplification of the perturbation velocity field.[11]

This paper describes the results of an exploratory study of the advantages obtained by combining IEM's with the method of parametric differentiation.[12] In this particular application of the method of parametric differentiation (MPD), the approach is distinctively different from the approach used by Norstrud.[5] In Norstrud's application of the MPD, the method is exploited to solve a system of nonlinear algebraic equations. Also, Norstrud's formulation retains the disputable assumption that the perturbation velocity is zero upstream and downstream of the airfoil.

In the present method, the nonlinear unsteady transonic flow equation for small perturbations is transformed into a linear equation with the use of the MPD. The linear equation is split into a pair of weakly coupled partial differential equations by writing the transformed perturbation potential as the sum of a steady component and an unsteady component. The solution of the steady equation as an integral equation is based on Ogana's[8] treatment. However, the numerical solution of the integral equation and the handling of the singularity of the integrand are done in an entirely different manner. As a test case, the formulation developed in this paper is applied to predict the steady transonic flow over a nonlifting parabolic-arc airfoil.

Analysis

The partial differential equation that governs transonic flow may be written as

$$[(1-M^2)-M^2(\gamma+1)\phi_x]\phi_{xx}+\phi_{yy}+\phi_{zz}$$
$$-2M^2\phi_{xt}-M^2\phi_{tt}=0 \tag{1}$$

with M as the freestream Mach number, and ϕ as the perturbation potential. Here, the flow is assumed to be unsteady, inviscid, and compressible past a thin wing at a small angle of attack. The wing lies in the x-y plane with the wind axis parallel to the x axis. The z axis coincides with the lift direction. The appropriate boundary conditions include the classical velocity tangency condition on the wing surface; the Kutta condition at the trailing edge, i.e., ϕ and its derivatives vanish approximately at infinity; and zero pressure difference across the plane $z=0$ at all regions outside of the wing.[11]

The MPD is not considered in detail here. Rubbert and Landahl[12] have described this method extensively. The thickness ratio ϵ of the wing section is used as the parameter for the parametric differentiation. The transformation of Eq. (1) to the ϵ-space with

$$g=\frac{\partial\phi}{\partial\epsilon} \quad \text{and} \quad u=\frac{\partial\phi}{\partial x} \tag{2}$$

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results in the linear equation

$$M^2 g_{tt} + 2M^2 g_{xt} - g_{yy} - g_{yy} - (1-M^2) g_{xx} = -M^2(\gamma+1)(ug_x)_x \tag{3}$$

The boundary conditions should also be transformed to the ϵ-space with the limitation that they be linear in g in order that the MPD be applicable.

Assume that $\bar{g}(x,y,z,t)$ may be written as the sum ofa steady component $\hat{g}(x,y,z)$ and an unsteady component $\tilde{g}(x,y,z,t)$ and assume that $\tilde{g} \ll \hat{g}$. Consistent with the latter assumption, the velocity component parallel to the x axis may be decomposed into a steady component $\hat{u}$ and an unsteady component $\tilde{u}$, where $\tilde{u} \ll \hat{u}$. These approximations simplify Eq. (3) to the following form[11]:

$$\hat{g}_{zz} + \hat{g}_{yy} + (1-M^2)\hat{g}_{xx} = M^2(\gamma+1)(\hat{u}\hat{g}_x)_x \tag{4}$$

$$M^2\tilde{g}_{tt} + 2M^2\tilde{g}_{xt} - \tilde{g}_{zz} - \tilde{g}_{yy} - (1-M^2)\tilde{g}_{xx} = -M^2(\gamma+1)(\hat{u}\tilde{g}_x)_x \tag{5}$$

Equations (4) and (5) for $\hat{g}$ and $\tilde{g}$ are weakly coupled, $\hat{u}$ being the coupling variable. We need not solve Eqs. (4) and (5) simultaneously, but must solve Eq. (4), to obtain $\hat{u}$, before solving Eq. (5).

We limit our focus to two-dimensional steady flows. Organa[8] gives the necessary integral equation formulation to solve Eq. (4). The complimentary solution corresponds to the classical linearized subsonic solution, while the particular solution is written in an integral form.

Green's theorem may be written as

$$\iint_S (\psi\nabla^2\Omega - \Omega\nabla^2\psi)\mathrm{d}S = -\int_C \left(\psi\frac{\partial\Omega}{\partial n} - \Omega\frac{\partial\psi}{\partial n}\right)\mathrm{d}C - \lim_{\sigma\to 0}\int_\sigma \left(\psi\frac{\partial\Omega}{\partial n} - \Omega\frac{\partial\psi}{\partial n}\right)\mathrm{d}C \tag{6}$$

where S is the region bounded by a sectionally smooth curve C; σ is a circular cavity surrounding a point P in S; Ω is a function with continuous first and second derivatives in S; Ψ is a function with continuous first and second derivatives in S and satisfies Laplace's equation, except possibly at the point P; and n is the inward normal to C. Equation (4) and the equation for Ψ may be combined to give[11]

$$\psi\nabla^2\hat{g} - \hat{g}\nabla^2\psi = \psi(\hat{u}\hat{g}_x)_x \tag{7}$$

Green's theorem [Eq. (6)] can be applied to the left-hand side of Eq. (7) if we let $\hat{g}$ correspond to Ω, i.e.,

$$-\int_C \left(\psi\frac{\partial\hat{g}}{\partial n} - \hat{g}\frac{\partial\psi}{\partial n}\right)\mathrm{d}i - \lim_{\sigma\to 0}\left[\int_\sigma \left(\psi\frac{\partial\hat{g}}{\partial n} - \hat{g}\frac{\partial\psi}{\partial n}\right)\mathrm{d}\sigma\right] = \iint_S \psi(\hat{u}\hat{g}_x)_x \mathrm{d}S \tag{8}$$

The contour C consists of segments C_∞, C_W, C_B, and Σ, where C_∞ is the contour at infinity C_W, that along the wake, C_B, and Σ over the body and the shock surface, respectively.

Now identify Ψ with the elementary solution of $\nabla^2\Psi = 0$, where

$$\psi(x-\xi, z-\zeta) = \ell n[(x-\xi)^2 + (2-\zeta^2)^2]^{1/2} \tag{9}$$

and where (ξ,ζ) is the observation point.

As shown in Sivaneri,[11] the integral equation for $\hat{g}_B(\xi,\zeta)$ takes the form

$$\hat{g}(\xi,\zeta) = \hat{g}_B(\xi,\zeta) + \frac{1}{2\pi}\int_\Sigma \psi\frac{\partial\hat{g}}{\partial n}\mathrm{d}\Sigma + \frac{1}{2\pi}\int_\Sigma \psi\hat{u}\hat{g}_x \mathrm{d}z - \frac{1}{2\pi}\iint_S \psi_x\hat{u}\hat{g}_x \mathrm{d}x\mathrm{d}z \tag{10}$$

The term $\hat{g}_B(\xi,\zeta)$ is the classical linearized subsonic solution. This term remains the same not only for all iterations, but also for all ϵ-levels. The double integral is the contribution due to the nonlinearities of the transonic equation. The integrals along the shock surface represent the jump in $\hat{u}$ and in the derivatives of $\hat{g}$ across the shock.

Numerical Procedure

Equation (10) for $\hat{g}(\xi,\zeta)$ reduces, in the absence of shocks, to

$$\hat{g}(\xi,\zeta) = \hat{g}_B(\xi,\zeta) - \frac{1}{2\pi}\iint_S \psi_x\hat{u}\hat{g}_x \mathrm{d}x\mathrm{d}z \tag{11}$$

At the initial ϵ-level, e.g., $\epsilon = 0.01$, $\hat{g} \approx \hat{g}_B$, where $\hat{g}_B$ is the solution of the linearized subsonic flow. At a higher ϵ-level, the presence of $\hat{g}_x$ and $\hat{u}$ on the right-hand side of Eq. (11) necessitates an iteration procedure to compute $\hat{g}(\xi,\zeta)$ at that ϵ-level. At a fixed ϵ-level, only $\hat{g}_x(x,z)$ is iterated upon, while $\hat{u}(x,z)$ is treated to be constant. At the end of the iteration procedure, $\hat{u}(x,z)$ is updated by solving Eq. (2). The Gauss-Seidel method is used to obtain a solution by iteration. $\hat{g}(x,z)$ at the previous ϵ-level is used to start the iteration on $\hat{g}_x(x,z)$ at the present level.

The integrand on the right side of Eq. (11) has a singularity at the observation point $x=\xi$ and $z=\zeta$. We have selected to treat this weak singularity by subtraction,[11] as previously proposed by Radbill.[13] The integral on the right side of Eq. (11) involves the numerical integration over a rectangular region and over a circular region (the singular integral component). We selected to resolve this apparent conflict by choosing a circular region circumscribing the rectangular region. The error introduced by this approximation is negligible, particularly when the rectangular region is large compared to the chord of the airfoil.[11]

We have selected to use a nonuniform mesh size in the z direction, in which direction no derivatives need to be computed. But, to accommodate Simpson's rule, have the first two segments be equal, say $\mathrm{d}z_1$, the next two be equal ($\mathrm{d}z_2$), and so on. In the positive x direction, consider three regions—the vicinity of the airfoil, a middle region, and the far field. The mesh size $\mathrm{d}x$ within each region is uniform but different for each region. A similar argument holds for the negative x direction. In choosing a differentiation scheme to compute $\hat{g}_x$ and ϕ_x, it has been decided to use the 5-point differentiation formulas based on Lagrangian interpolation. The problem of maintaining accuracy in the results near the leading and trailing edges of the airfoil has not been treated in detail by the authors. Due to the lack of bluntness and zero incidence of the test cases considered, these singularities may be considered to be weak.[3,14] By exploiting the nonuniform mesh as previously described, and by computing around these singularities, reasonable results have been obtained.[11]

Results and Conclusions

As a test case, we apply the preceding formulation to a nonlifting parabolic-arc airfoil. For this flow, which has no imbedded shock waves, $\hat{g}(x,z)$ is symmetric about the x axis and antisymmetric about the z axis. This feature enables us to restrict our attention to only one-quarter of the flowfield. Referring to Eq. (11), $\hat{g}(\xi,\zeta)$ is to be computed for (ξ,ζ) in

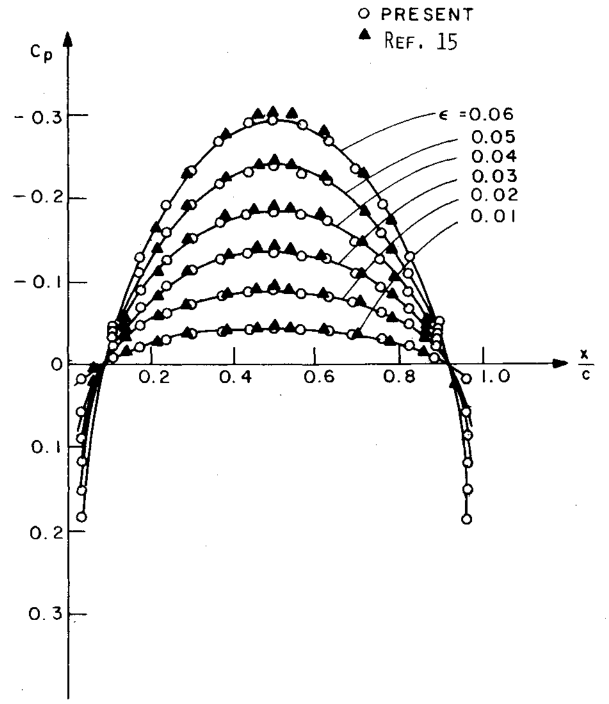

Fig. 1 Surface pressure coefficients for a parabolic-arc airfoil at $M = 0.825$.

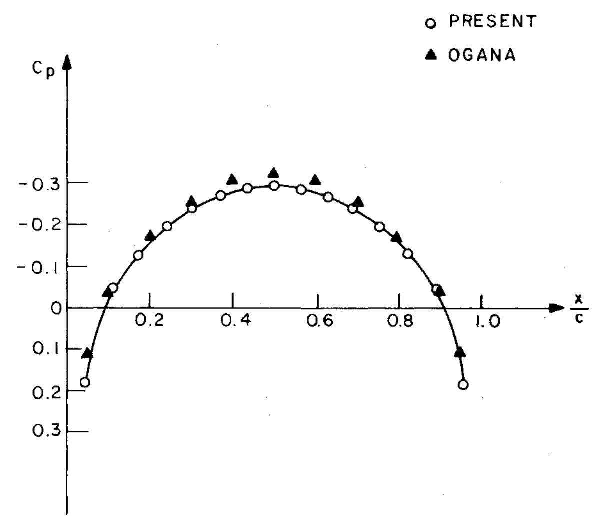

Fig. 2 Surface pressure coefficient for a parabolic-arc airfoil at $\epsilon = 0.06$ and $M = 0.825$.

the one-quarter plane. But, still the integration in Eq. (11) is to be carried over the entire region, and this is achieved through the symmetry and/or the antisymmetry of Ψ_x, $\hat{u}$, and $\hat{g}_x$ about the coordinate axes. Thus, the computation time is considerably reduced.

The calculations have been completed for freestream Mach numbers of 0.806 and 0.825. To keep the flow subcritical, the computations were done up to a thickness ratio (ϵ) of 0.06 starting from $\epsilon = 0.01$ and increasing ϵ in steps of 0.01. To return to the original ϕ-space, we have to numerically solve Eq. (2) for $\phi(x,z)$. We need to know $\phi(x,z)$ at the initial ϵ-level ($\epsilon = 0.01$) to solve Eq. (2) for $\phi(x,z)$ at higher ϵ-levels. The solution of the classical linearized subsonic equation was used as an initial solution. Equation (2) was solved using the trapezoidal rule. It took a maximum of eight iterations to converge to a tolerance of 0.00001 with fewer iterations at lower ϵ-levels.

After solving Eq. (2) for $\phi(x,z)$, the linearized pressure coefficient C_P was computed.[11] Typical results are presented in Figs. 1 and 2. Figure 1 compares the results of the present method for thickness ratios of 0.01-0.06 with Whitlow and Harris[15] for freestream Mach number 0.825. Figure 2 compares C_P at $M = 0.825$ and $\epsilon = 0.06$ with Ogana.[8]

Whitlow and Harris[15] combine finite differences with parametric differentiation to solve the transonic problem. Referring to Fig. 1, the difference between the present results and those of Whitlow and Harris[15] is more pronounced near the peak of the C_P curve. This difference varies from 3% at $\epsilon = 0.02$ to 6% at $\epsilon = 0.06$. Whitlow and Harris used a predictor-corrector method to solve for C_P, which requires the storage of the ϕ-field at three previous ϵ-levels. But, as mentioned earlier, we used a simpler but less accurate scheme—the trapezoidal rule. Thus, the difference in results may be attributed to the different integration methods used.

The current effort is the first to combine an IEM and the MPD to solve a transonic flow problem in which the MPD exploits a physical parameter characterizing the flow. Extensions of this formulation to unsteady flows and lifting airfoils including supercritical flows are in progress.

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